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On the completeness of coherent states generated by binomial distribution

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Abstract. We demonstrate the completeness of boson- and spin-coherent states generated by binomial distribution by furnishing the exact analytical form of their respective positive weight functions in resolution of unity. These weight functions are solutions of the associated truncated Stieltjes power–moment problem. We elaborate on the non-unique character of these solutions.

1. Introduction

Solving a quantum mechanical problem often amounts to forming particular linear combinations of eigenfunctions of a Hermitian operator. The coefficients of this linear combination are then parametrized by a limited number of parameters whose choice optimizes the solution. The standard normalized coherent states $|z\rangle$ [1] are formed by a special infinite combination of eigenstates of a linear harmonic oscillator $\hat{H}_0 = \hat{a}^\dagger \hat{a}$, denoted by $|n\rangle$ ($[\hat{a}, \hat{a}^\dagger] = 1$, $\hat{H}_0|n\rangle = n|n\rangle$, $\langle n|n'\rangle = \delta_{n,n'}$, $n = 0, 1, \ldots, \infty$), such that all the coefficients are parametrized by a single complex number z:

$$|z\rangle = \exp\left(-\frac{|z|^2}{2}\right)\exp(z\hat{a}^\dagger)|0\rangle \tag{1}$$

$$= \exp\left(-\frac{|z|^2}{2}\right)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle \tag{2}$$

where in equation (1), $|0\rangle$ is the ground state of $\hat{H}_0$ (the vacuum).

Likewise, if we consider a single spin S interacting with the magnetic field $\vec{H} = (0, 0, h)$, $h > 0$, through $\hat{H}_S = -h\hat{S}_z$ then we can use, for complex μ, the eigenstates $|p\rangle$ of $\hat{H}_S$ ($\hat{S}_z|p\rangle = (S - p)|p\rangle$, $\langle p|p'\rangle = \delta_{p,p'}$, $0 \leqslant p \leqslant 2S$) to construct the normalized spin-coherent state [2]:

$$|\mu\rangle = \mathcal{N}^{-\frac{1}{2}}(|\mu|^2)\exp(\mu\hat{S}_-)|0\rangle \tag{3}$$

$$= \frac{1}{(1+|\mu|^2)^S}\sum_{p=0}^{2S}\binom{2S}{p}^{\frac{1}{2}}\mu^p|p\rangle \tag{4}$$

where in equation (3), $|0\rangle$ is the ground state of $\hat{H}_S$ with $\vec{\hat{S}} \parallel \vec{H}$. In general, for $z \neq z'$, $\mu \neq \mu'$, $\langle z|z'\rangle \neq 0 \neq \langle\mu|\mu'\rangle$, with one exception: if $\mu' = -1/\mu^*$, then $\langle\mu|\mu'\rangle = 0$.

The parametrization in terms of a single complex number z in equation (1) and μ in equation (3) permits one to demonstrate that $|z\rangle$ and $|\mu\rangle$ form complete sets, i.e. they allow the resolution of unity in their respective state spaces where

$$I_b = \sum_{n=0}^{\infty} |n\rangle\langle n| \tag{5}$$

and

$$I_{2S} = \sum_{p=0}^{2S} |p\rangle\langle p| \tag{6}$$

hold. The resolution of unity in terms of $|z\rangle$ and $|\mu\rangle$ can be achieved by finding positive weight functions $W_b(|z|^2)$ and $V_{2S}(|\mu|^2)$ such that, for $\mathrm{d}^2\lambda \equiv \mathrm{d}(\mathrm{Re}\,\lambda)\,\mathrm{d}(\mathrm{Im}\,\lambda)$, $\lambda = z, \mu$, the following equations are valid:

$$\int\!\!\int_{\mathbb{C}} \mathrm{d}^2 z |z\rangle W_b(|z|^2)\langle z| = I_b \tag{7}$$

$$\int\!\!\int_{\mathbb{C}} \mathrm{d}^2 \mu |\mu\rangle V_{2S}(|\mu|^2)\langle \mu| = I_{2S}. \tag{8}$$

The functions $W_b(|z|^2)$ and $V_{2S}(|\mu|^2)$ are known [1,2]:

$$W_b(|z|^2) = \frac{1}{\pi} = \mathrm{const} \tag{9a}$$

$$V_{2S}(|\mu|^2) = \frac{2S+1}{\pi}(1+|\mu|^2)^{-2}. \tag{9b}$$

The existence of the resolution of unity is an important ingredient in the general definition of coherent states [3] as it restricts very seriously possible function sets. Moreover, the weight function intervenes directly when the coherent states are used in variational treatment of thermodynamics via Lieb–Berezin inequalities [4].

In spite of the huge literature on coherent states and their applications [1,5], there are only a few examples of group-generated coherent states for which the weight function is actually explicitly known [6–11]. The situation is even less satisfactory for the states generated without any underlying group structure, i.e. without using the exponential function at all. Recently progress in this direction has been achieved by finding a positive weight function for coherent states generated by generalized exponential functions [12], and by some special functions [13].

In this work we shall consider the boson and spin states generated by binomial distributions and demonstrate their completeness.

2. Binomial boson state

The binomial coherent states (BS) were introduced in [14]. They are finite linear combinations of the number states $|n\rangle$ of the harmonic oscillator Hamiltonian $\hat{H}_0$. When parametrized in terms of complex $y = r\mathrm{e}^{\mathrm{i}\delta}$, the normalized BS, for $\eta = \sin^2 r$, for $|0\rangle$ vacuum of $\hat{H}_0$ reads [15]:

$$|y; M\rangle = \sum_{n=0}^{M} \left[\binom{M}{n}\eta^n(1-\eta)^{M-n}\right]^{\frac{1}{2}} \frac{\mathrm{e}^{-\mathrm{i}(\delta+\pi)n}}{\sqrt{n!}}(a^\dagger)^n|0\rangle \qquad 1 \leqslant M < \infty. \tag{10}$$

As the $su(2)$ algebra involved here is the same as in the case of spin-coherent states $|\mu\rangle$ (see, equation (4)), then the resolution of identity for $|y; M\rangle$ is realized with equation (9b) for $2S = M$ and $|\mu|^2 = r^2$.

New kinds of binomial coherent states were introduced recently in [16]. They are defined for complex λ, integer M and $|0\rangle$ the vacuum of $\hat{H}_0$, as

$$|\lambda; M\rangle = \mathcal{N}_M^{-\frac{1}{2}}(|\lambda|^2)(\hat{a}^\dagger + \lambda)^M|0\rangle \qquad 1 \leqslant M < \infty. \tag{11}$$

Many expectation values of physical quantitites in these states were calculated and their relevance to the experiments discussed [16]. Evidently these states contain contributions of the first M states of the harmonic oscillator. We will consider now the problem of completeness of $|\lambda; M\rangle$. Following [16], we expand the binomial and using $(\hat{a}^\dagger)^n|0\rangle = \sqrt{n!}|n\rangle$ we rewrite equation (11) as

$$|\lambda; M\rangle = \mathcal{N}_M^{-\frac{1}{2}}(|\lambda|^2)\sum_{n=0}^{M}\frac{M!}{(M-n)!\sqrt{n!}}\lambda^{M-n}|n\rangle \tag{12}$$

from which the normalization $\mathcal{N}_M(|\lambda|^2)$ follows:

$$\begin{aligned}\mathcal{N}_M(|\lambda|^2) &= M!\sum_{n=0}^{M}\frac{M!}{(M-n)!n!}\frac{1}{n!}|\lambda|^{2n}\\ &= M!L_M(-|\lambda|^2)\end{aligned} \tag{13}$$

where $L_M(x)$ is the Mth Laguerre polynomial,

$$L_M(x) = \sum_{n=0}^{M}(-1)^n\frac{1}{n!}\binom{M}{M-n}x^n. \tag{14}$$

We observe that $|\lambda; M\rangle$ is normalizable only for finite M, otherwise for $M \to \infty$ and $|\lambda| = \text{const}$ $\mathcal{N}_M(|\lambda|^2)$ diverges. We shall now require the resolution of unity in the subspace of $M + 1$ states:

$$\int\!\!\int_{\mathbb{C}} \mathrm{d}^2\lambda|\lambda; M\rangle W_M(|\lambda|^2)\langle\lambda; M| = \sum_{n=0}^{M}|n\rangle\langle n| = I_M \tag{15}$$

i.e. we attempt to find a positive $W_M(|\lambda|^2)$ satisfying equation (15). Substitute equations (12) and (13) into equation (15), use $\mathrm{d}^2\lambda = \frac{1}{2}\,\mathrm{d}(|\lambda|^2)\,\mathrm{d}\theta$, $(\lambda = |\lambda|\mathrm{e}^{\mathrm{i}\theta})$, and perform the angular integration. Then we obtain for $|\lambda|^2 \equiv x$:

$$\pi\sum_{n=0}^{M}|n\rangle\langle n|\left[\frac{M!}{((M-n)!)^2 n!}\int_0^\infty x^{M-n}\frac{W_M(x)}{L_M(-x)}\,\mathrm{d}x\right] = I_M \tag{16}$$

or the conditions on $W_M(x)$

$$\pi\int_0^\infty x^{M-n}\left[\frac{W_M(x)}{L_M(-x)}\right]\mathrm{d}x = \frac{((M-n)!)^2 n!}{M!} \qquad n = 0, 1, \ldots, M \tag{17}$$

which upon redefining $n \to M - n$, become

$$\pi\int_0^\infty x^n\tilde{W}_M(x)\,\mathrm{d}x = \frac{(n!)^2(M-n)!}{M!} \qquad n = 0, 1, \ldots, M \tag{18}$$

which is the truncated Stieltjes power–moment problem [17] for the unknown function $\tilde{W}_M(x) = W_M(x)/L_M(-x)$. The difficult problem of proving the existence of the solution $\tilde{W}_M(x) > 0$ in equation (18) will be overcome by a construction of such a solution. We will interpret equation (18) as the inverse Mellin transform problem [18] as done in related contexts before [12, 13, 19], by setting for complex s, $n \to s - 1$ and rewriting equation (18) as

$$\pi\int_0^\infty x^{s-1}\tilde{W}_M(x)\,\mathrm{d}x = \frac{\Gamma^2(s)\Gamma(M+2-s)}{M!} \equiv \pi\mathcal{M}[\tilde{W}_M(x); s] \qquad (\mathrm{Re}\, s < M + 2). \tag{19}$$

In other words

$$\pi \tilde{W}_M(x) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\Gamma^2(s)\Gamma(M+2-s)}{M!} x^{-s}\, \mathrm{d}s \tag{20}$$

$$= \mathcal{M}^{-1}\left[\frac{\Gamma^2(s)\Gamma(M+2-s)}{M!}; x\right] \qquad (\mathrm{Re}\, s < M+2). \tag{21}$$

In equations (19) and (21) $f^*(s) = \mathcal{M}[f(x); s]$ is the Mellin transform of $f(x)$ and $f(x) = \mathcal{M}^{-1}[f^*(s); x]$ is the inverse Mellin transform of $f^*(s)$ [18]. The complex integration in equation (20) can be avoided if one uses the Mellin convolution property of inverse Mellin transforms [20] (also called the generalized Parseval formula), which for arbitrary $f^*(s)$ and $g^*(s)$, for which $f(x)$ and $g(x)$ exist, states that

$$\int_0^\infty f\left(\frac{x}{t}\right) g(t) \frac{1}{t}\, \mathrm{d}t = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} f^*(s) g^*(s) x^{-s}\, \mathrm{d}s. \tag{22}$$

In applying equation (22) to (20) we choose $f^*(s) \equiv \Gamma(M+2-s)\Gamma(s)$ with $\mathrm{Re}\, s < M+2$, which identifies $f(x)$ as

$$\mathcal{M}^{-1}[f^*(s); x] = (M+1)!/(1+x)^{M+2} \tag{23}$$

and $g^*(s) \equiv \Gamma(s)/M!$ which results in $g(x) = \mathrm{e}^{-x}/M!$, and obtain

$$\tilde{W}_M(x) = \frac{(M+1)!}{\pi M!} \int_0^\infty \frac{\mathrm{e}^{-t}}{(1+\frac{x}{t})^{M+2}} \frac{1}{t}\, \mathrm{d}t \tag{24}$$

$$= \frac{(M+1)}{\pi} \int_0^\infty \frac{t^{M+1}\mathrm{e}^{-t}}{(t+x)^{M+2}}\, \mathrm{d}t \tag{25}$$

which is a positive function of x. We call it the principal solution. The integral in equation (25) can be expressed in terms of known functions in several ways. Using the formula (6.10) of [20], equation (25) can be written in terms of Tricomi's integral $\Psi(a; c; z)$ as

$$\tilde{W}_M(x) = \frac{(M+1)}{\pi} \Gamma(M+2)\Psi(M+2; 1; x) \tag{26}$$

where

$$\Psi(a; c; z) = \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-zt} t^{a-1} (1+t)^{c-a-1}\, \mathrm{d}t \qquad (\mathrm{Re}\, a > 0, \mathrm{Re}\, z > 0) \tag{27}$$

is the integral representation of the confluent hypergeometric function of Tricomi, $\Psi(a; c; z)$. Another form of equation (25) is obtained if we change the variable $y = t + x$, expand the binomial $(y - x)^{M+1}$ in the resulting integrand and integrate term by term. The equation obtained is

$$\tilde{W}_M(x) = \frac{W_M(x)}{L_M(-x)} = \frac{M+1}{\pi} \mathrm{e}^x \sum_{n=0}^{M+1} (-1)^n \binom{M+1}{n} \mathrm{E}_{n+1}(x) \tag{28}$$

where $\mathrm{E}_k(x)$ is the generalized exponential integral of order k defined by [22]:

$$\mathrm{E}_k(x) \stackrel{\mathrm{def}}{=} \int_1^\infty \frac{\mathrm{e}^{-xt}}{t^k}\, \mathrm{d}t \qquad k = 0, 1, 2, \ldots \qquad \mathrm{Re}\, x > 0. \tag{29}$$

In figure 1 we have presented the weight functions $W_M(x)$ for $M = 2, 4, 6, 10$. They are positive for all $x > 0$ and thereby we conclude that $|\lambda; M\rangle$ of equation (11) are a complete set. In figure 2 we have presented the functions $\tilde{W}_M(x)$, the principal solutions of equation (18).

Note that the solutions of a truncated moment problem are never unique. A minimal set of additional solutions is obtained by adding to the principal solution a function whose exactly $M + 1$ first moments vanish, under the condition that the resulting sum be strictly

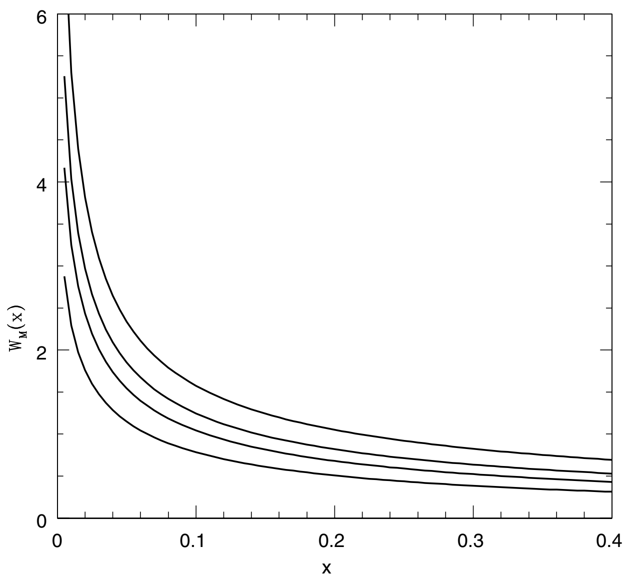

Figure 1. $W_M(x)$ versus x for $M = 2, 4, 6, 10$. The lowest curve corresponds to $M = 2$, the two higher correspond to $M = 4$ and 6, and the highest curve corresponds to $M = 10$.

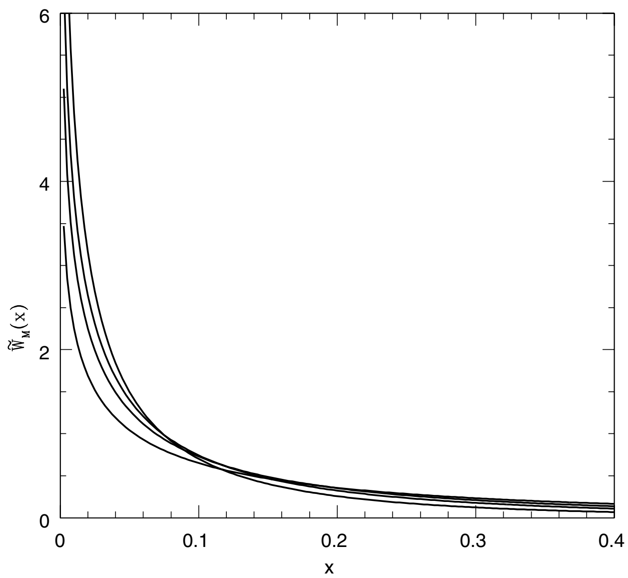

Figure 2. $\tilde{W}_M(x)$ versus x for $M = 2, 4, 6, 10$. For $x < 0.1$, the lowest curve corresponds to $M = 2$, the two higher correspond to $M = 4$ and 6, and the highest curve corresponds to $M = 10$.

positive. In a separate publication we have developed general methods to produce many distinct families of functions with first $M + 1$ moments vanishing [21]. We quote one of them here: $\phi_{M+1}(x) = \mathrm{e}^{-x} L_{M+1}(x)$, where $L_M(x)$ is the Laguerre polynomial. Consequently an infinite one-parameter family of additional solutions of equation (18) are given by

$$\tilde{W}_M(x;\kappa) = \tilde{W}_M(x)\left(1+\kappa\frac{\mathrm{e}^{-x}L_{M+1}(x)}{\tilde{W}_M(x)}\right) \tag{30}$$

where $\kappa < \kappa_M$ to assure the positivity of $\tilde{W}_M(x;\kappa)$: $\kappa_1 = \frac{1}{8}$, $\kappa_2 = \frac{1}{14}$, $\kappa_3 = \frac{1}{44}$, $\kappa_4 = \frac{1}{88}, \ldots$ etc.

3. Binomial spin-coherent state

We now construct the analogy of the spin-coherent state of equation (3) using a power of a binomial instead of an exponential. Consider a single particle of spin S, ($S = \frac{1}{2}, 1, \frac{3}{2}, \ldots$), and the state $|0\rangle$, the ground state of $\hat{H}_S$, defined as $\hat{S}_z|0\rangle = S|0\rangle$. The spin-lowering operator $\hat{S}_-$ is nilpotent of order $2S+1$, i.e. $(\hat{S}_-)^{2S+1} \equiv 0$. Choose a complex μ and define the state $|\mu;2S\rangle$ as

$$|\mu;2S\rangle = \mathcal{N}_{2S}^{-\frac{1}{2}}(|\mu|^2)(\hat{S}_- + \mu)^{2S}|0\rangle \tag{31}$$

$$= \mathcal{N}_{2S}^{-\frac{1}{2}}(|\mu|^2)\sum_{p=0}^{2S}\binom{2S}{p}\mu^{2S-p}(\hat{S}_-)^p|0\rangle \tag{32}$$

which with

$$(\hat{S}_-)^p|0\rangle = \left[\frac{p!(2S)!}{(2S-p)!}\right]^{\frac{1}{2}}|p\rangle \tag{33}$$

where $|p\rangle$ satisfy

$$\hat{S}_z|p\rangle = (S-p)|p\rangle \qquad 0 \leqslant p \leqslant 2S \tag{34}$$

results in

$$|\mu;2S\rangle = \mathcal{N}_{2S}^{-\frac{1}{2}}(|\mu|^2)\sum_{p=0}^{2S}\left[\frac{(2S)!}{(2S-p)!}\right]^{\frac{3}{2}}\frac{1}{(p!)^{\frac{1}{2}}}\mu^{2S-p}|p\rangle \tag{35}$$

$$= \mathcal{N}_{2S}^{-\frac{1}{2}}(|\mu|^2)\sum_{p=0}^{2S}p!\binom{2S}{p}^{\frac{3}{2}}\mu^{2S-p}|p\rangle. \tag{36}$$

If $\langle\mu;2S|\mu;2S\rangle = 1$ then for $|\mu|^2 \equiv x$

$$\mathcal{N}_{2S}(x) = \sum_{p=0}^{2S}[(2S-p)!]^2\binom{2S}{p}^3 x^p \tag{37}$$

$$= ((2S)!)^2 R_{2S}(x) \tag{38}$$

where the polynomial $R_{2S}(x)$ defined in (38) as

$$R_{2S}(x) = \sum_{p=0}^{2S}\frac{1}{(p!)^2}\binom{2S}{p}x^p \tag{39}$$

can be expressed through the generalized hypergeometric function as

$$R_{2S}(x) = {}_1F_2(-2S;1,1;-x). \tag{40}$$

Using exclusively the angular momentum commutation relations, a matrix element of a general function $\langle\mu;2S|g(\hat{S}_+,\hat{S}_-,\hat{S}_z)|\mu;2S\rangle$ can be calculated in a closed form. We quote one example here:

$$\langle\mu; 2S|\hat{S}_{-}|\mu; 2S\rangle = \frac{\mu}{\mathcal{N}_{2S}(|\mu|^2)} \times \sum_{p=1}^{2S} p!(p-1)!\left[\binom{2S}{p-1}\binom{2S}{p}\right]^{\frac{3}{2}} \sqrt{p(2S+1-p)}|\mu|^{2(2S-p)}. \tag{41}$$

We now require the resolution of unity in terms of states $|\mu; 2S\rangle$ of equation (31) seeking $W_{2S}(|\mu|^2) > 0$ such that

$$\iint_{\mathbb{C}} \mathrm{d}^2\mu|\mu; 2S\rangle W_{2S}(|\mu|^2)\langle\mu; 2S| = \sum_{p=0}^{2S} |p\rangle\langle p| = I_{2S}. \tag{42}$$

Through corresponding modifications of the considerations leading to equation (16), equation (42), using equations (35) and (38), transforms into ($|\mu|^2 \equiv x$):

$$\pi \sum_{p=0}^{2S} |p\rangle\langle p| \left[\frac{p!\binom{2S}{p}^3}{(2S)!^2} \int_0^\infty x^{2S-p}\left[\frac{W_{2S}(x)}{R_{2S}(x)}\right]\mathrm{d}x\right] = I_{2S} \tag{43}$$

resulting in the following truncated Stieltjes power–moment problem for an unknown function: $\tilde{W}_{2S}(x) = W_{2S}(x)/R_{2S}(x)$:

$$\pi\int_0^\infty x^p \tilde{W}_{2S}(x)\,\mathrm{d}x = \frac{(2S-p)!(p!)^3}{(2S)!} \qquad p = 0, 1, \ldots, 2S. \tag{44}$$

We will obtain $\tilde{W}_{2S}(x)$ again using the Mellin convolution technique and call it the principal solution. Setting $p \to s-1$ for complex s, equation (44) takes the form

$$\pi\int_0^\infty x^{s-1}\tilde{W}_{2S}(x)\,\mathrm{d}x = \frac{\Gamma(2S+2-s)\Gamma(s)^3}{(2S)!} \qquad \mathrm{Re}\, s < 2S+2. \tag{45}$$

Use equation (22) again with $f^*(s) = \Gamma(2S+2-s)\Gamma(s)$ with $\mathrm{Re}\, s < 2S+2$, and $g^*(s) = \Gamma^2(s)/(2S)! \equiv g_1^*(s)\cdot g_1^*(s)/(2S)!$ where $g_1(x) = \mathrm{e}^{-x}$. We once again apply equation (22) to $g_1^*(s)\cdot g_1^*(s)$ and observe that $g(x) = 2K_0(2\sqrt{x})/(2S)!$, where $K_0(y)$ is the modified Bessel function of the second kind (or the MacDonald function), as a consequence of the Sommerfeld representation of $K_\nu(y)$:

$$K_\nu(y) = \frac{1}{2}\left(\frac{y}{2}\right)^\nu \int_0^\infty \frac{\mathrm{e}^{-t-\frac{y^2}{4t}}}{t^{\nu+1}}\,\mathrm{d}t. \tag{46}$$

Now perform the Mellin convolution of $f^*(s)$ and $g^*(s)$ and get

$$\pi(2S)!\tilde{W}_{2S}(x) = 2\Gamma(2S+2)\int_0^\infty \frac{K_0(2\sqrt{t})}{(1+\frac{x}{t})^{2S+2}}\frac{1}{t}\,\mathrm{d}t \tag{47}$$

and

$$\tilde{W}_{2S}(x) = \frac{4(2S+1)}{\pi}\int_0^\infty \frac{y^{4S+3}K_0(2y)}{(y^2+x)^{2S+2}}\,\mathrm{d}y \tag{48}$$

which is a positive function of x, establishing the completeness of $|\mu; 2S\rangle$. The integral in equation (48) can be expressed in terms of the infinite series of gamma and polygamma functions:

$$\tilde{W}_{2S}(x) = \frac{1}{2\pi}\frac{1}{(2S)!}\sum_{k=0}^\infty x^k \frac{\Gamma(2S+2+k)}{\Gamma^3(k+1)}[2\ln(x)(3\psi(1+k) - \psi(2S+2+k)) - \ln^2(x) - \pi^2 - \psi^2(2S+2+k) - \psi^{(1)}(2S+2+k) - 9\psi^2(1+k) + 3\psi^{(1)}(1+k) + 6\psi(2S+2+k)\psi(1+k)] \tag{49}$$

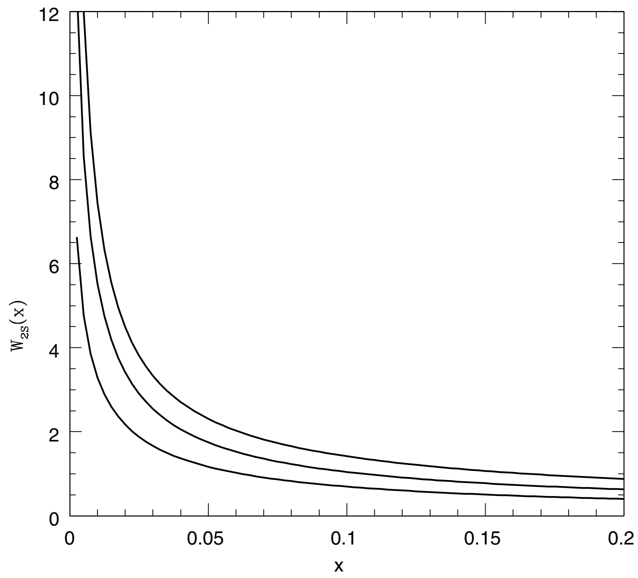

Figure 3. $W_{2S}(x)$ versus x for $S = 1, \frac{9}{2}, 12$. The lowest curve corresponds to $S = 1$, the middle curve to $S = \frac{9}{2}$, and the highest curve to $S = 12$.

where $\psi(t)$ is the digamma function, and $\psi^{(1)}(t)$ is the polygamma function of order one, compare [23].

Note in passing, that equation (49) can also be cast in a more compact form using the Meijer G-function [24]:

$$\tilde{W}_{2S}(x) = \frac{1}{\pi(2S)!} G^{3,1}_{1,3}\left(x \left| \begin{matrix} & -2S-1 & \\ 0, & 0, & 0 \end{matrix} \right. \right). \tag{50}$$

The graphical representation of $W_{2S}(x)$ is given in figure 3, and the graphical representation of $\tilde{W}_{2S}(x)$ is given in figure 4, for $2S = 2, 9, 24$.

In full analogy with equation (30) the additional minimal set of solutions of equation (44) is the one-parameter family

$$\tilde{W}_{2S}(x;\rho) = \tilde{W}_{2S}(x)\left(1 + \rho\frac{\mathrm{e}^{-x}L_{2S+1}(x)}{\tilde{W}_{2S}(x)}\right) \tag{51}$$

where $\rho < \rho_{2S}$ to assure the positivity of $\tilde{W}_{2S}(x;\rho)$: $\rho_1 = \frac{1}{10}$, $\rho_2 = \frac{1}{10}$, $\rho_3 = \frac{1}{25}$, $\rho_4 = \frac{1}{42}\ldots$, etc.

4. Discussion and conclusions

Coherent states are an extremely efficient and flexible tool to treat very different aspects of quantum problems. They permit one to analyse the ground state as well as the semiclassical limit involving high-quantum number excited states. The minimal set of defining conditions for coherent states, as expressed in [1, 3] is: (a) normalizability, (b) continuity in the label (which for both cases discussed in this work is an arbitrary complex number), (c) the resolution of unity, or completeness. The actual key for utility of coherent states is their property (c) which allows one to use them as a (usually non-orthogonal) basis.

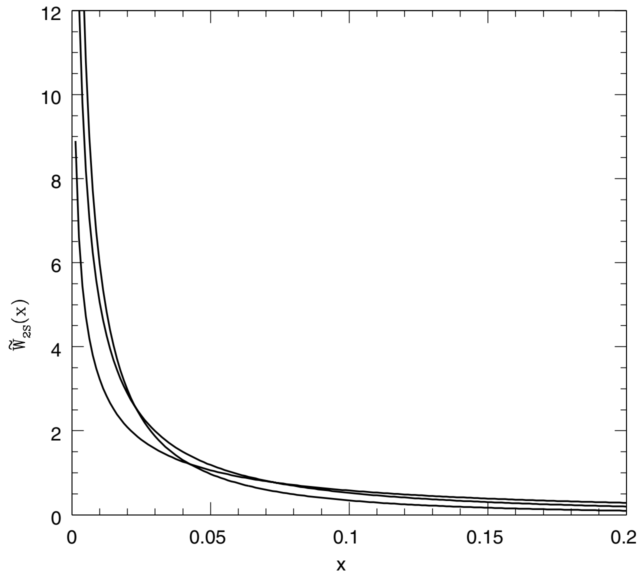

Figure 4. $\tilde{W}_{2S}(x)$ versus x for $S = 1, \frac{9}{2}, 12$. For $x > 0.1$, the highest curve corresponds to $S = 1$, the middle curve to $S = \frac{9}{2}$, and the lowest curve to $S = 12$.

In this work we have extended the construction of coherent states to include two new families which are generated by a binomial distribution instead of an exponential (as in case of standard boson- and spin-coherent states). Whereas (a) and (b) are satisfied here automatically (the states are polynomial in nature), point (c) needed special attention. The new feature of these states is that they are linear combinations of a finite number of Fock states, for both the spin and boson cases. The weight functions in the resolution of unity have been obtained by solving the truncated Stieltjes moment problem. To this end we have adapted to the truncated case the method of inverse Mellin transform developed previously for the full Stieltjes problem [12,13]. Whereas in the spin case the solution of Radcliffe [2], see equation (9*b*), already involved the truncated problem, the truncated boson problem appears to be new.

The common feature of the truncated moment problems, such as those in equations (18) and (44), is the non-unique character of their solutions. We have derived and presented in equations (30) and (51) the one-parameter minimal sets of their respective solutions. We still owe the reader the clarification of the term ‘minimal’. By this term we mean that in producing positive weight functions we have added to the principal solutions (which happen to be monotonic functions) a function whose exactly $M + 1$ first moments vanish, i.e $\phi_{M+1}(x) = e^{-x} L_{M+1}(x)$ where $L_n(x)$ is Laguerre polynomial. A more general set of solutions is obtained by adding a linear combination of functions having first $M + 1 + r$, $(r = 0, 1, 2, \ldots, k)$ moments vanishing, that is in context of equation (30), forming a multi-parameter family

$$\tilde{W}_M(x; \kappa_0, \kappa_1, \kappa_2, \ldots, \kappa_k) = \tilde{W}_M(x) \left(1 + \frac{e^{-x} \sum_{r=0}^{k} \kappa_r L_{M+1+r}(x)}{\tilde{W}_M(x)}\right). \tag{52}$$

Note that in equation (52) k may formally be equal to infinity. The parameters $\{\kappa_l\}_{l=0,1,2,\ldots,k}$ have to be so chosen as to assure the positivity of $\tilde{W}_M(x; \{\kappa_l\}_{l=0,1,2,\ldots,k})$. However, the $\phi_{M+1}(x)$ constitute just one subset of an infinite set of various functions whose first $M + 1$ moments vanish [21], so that an infinity of further generalizations of equation (52) are possible. They include non-analytic and singular functions, too. In any case the resulting huge arbitrariness in the choice of the weight function is a real problem. One can conceive certain criteria of either mathematical or of physical nature, for choosing the 'best' weight function. This challenging and still not entirely resolved question is under our active consideration.

The novel coherent wavefunctions proposed in this work are particularly well suited to tackle the problems with the energy cut-off, i.e. the situations where a finite number of Fock states is sufficient to describe the physics. Several aspects of this approach have been reviewed recently [25]. In a parallel development, the truncated coherent states appear naturally in the theory of field–atom interactions in the highly nonlinear media in the cavity [16, 26]. The method exposed in this work permits one to establish the completeness properties of many truncated states. Other important variants of coherent states, such as the photon-added [27, 28] and photon-substracted states [29] can be also successfully analysed by this method.

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References

[1] Klauder J R and Skagerstam B-S 1985 *Coherent States, Applications in Physics and Mathematical Physics* (Singapore: World Scientific)
[2] Radcliffe J M 1971 *J. Phys. A: Math. Gen.* **4** 313
[3] Klauder J R 1963 *J. Math. Phys.* **4** 1058
[4] Lieb E H 1994 *Proc. Symp. on Coherent States: Past, Present and Future (Oak Ridge)* ed D H Feng and J R Klauder (Singapore: World Scientific) p 267
[5] Zhang W M, Feng D H and Gilmore R 1990 *Rev. Mod. Phys.* **62** 867
[6] Barut A O and Girardello L 1971 *Commun. Math. Phys* **21** 41
[7] Bhaumik D, Bhaumik K and Dutta-Roy B 1976 *J. Phys. A: Math. Gen.* **9** 1507
[8] Perelomov A M 1972 *Commun. Math. Phys.* **26** 22
[9] Basu D 1992 *J. Math. Phys.* **33** 114
[10] Gazeau J P and Klauder J R 1999 *J. Phys. A: Math. Gen.* **32** 123
[11] Chaturvedi S 1996 *Mod. Phys. Lett.* A **35** 2805
[12] Penson K A and Solomon A I 1999 *J. Math. Phys.* **40** 2354
[13] Sixdeniers J-M, Penson K A and Solomon A I 1999 *J. Phys. A: Math. Gen.* **32** 7543
[14] Stoler D, Saleh B E A and Teich M C 1985 *Opt. Acta* **32** 345
[15] Fu H C and Sasaki R 1998 *J. Phys. A: Math. Gen.* **31** 901
[16] Fu H C, Feng Y Q and Solomon A I 2000 *J. Phys. A: Math. Gen.* **33** 2231
[17] Akhiezer N I 1965 *The Classical Moment Problem and Some Related Questions in Analysis* (London: Oliver and Boyd)
[18] Sneddon I N 1974 *The Use of Integral Transforms* (New York: McGraw-Hill)
[19] Fernandez D J, Hussin V and Nieto L M 1994 *J. Phys. A: Math. Gen.* **27** 3547
[20] Marichev O I 1983 *Handbook of Integral Transforms of Higher Transcendental Functions: Theory and Algorithmic Tables* (Chichester: Ellis Horwood)
[21] Sixdeniers J-M, Penson K A and Haddad L 2000 in preparation
[22] Abramovitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (New York: Dover) chapter 5
[23] Abramovitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (New York: Dover) chapter 6
[24] Prudnikov A P, Brychkov Yu A and Marichev O I 1990 *Integrals and Series, More Special Functions* vol 3 (New York: Gordon and Breach) p 617
[25] Roy B and Roy P 1998 *J. Phys. A: Math. Gen.* **31** 1307
[26] Leonski W 1997 *Phys. Rev.* A **55** 3874
[27] Agarwal G S and Tara K 1991 *Phys. Rev.* A **43** 492
[28] Dodonov V V, Marchiolli M A, Korennoy Ya A, Man'ko V I and Moukhin Y A 1998 *Phys. Rev.* A **58** 4087
[29] Dakna M, Anhut T, Opatrný T, Knöll L and Welsch D G 1997 *Phys. Rev.* A **55** 3184